OLLSCOIL NA hÉIREANN, CORCAIGH
THE NATIONAL UNIVERSITY OF IRELAND, CORK

COLÁISTE NA hOLLSCOILE, CORCAIGH
UNIVERSITY COLLEGE, CORK

TEAM MATH FINAL 2008

1. The lines

$$y = k^2 x + 12 \quad \text{and} \quad 2ky = 4x + 5, \quad (k \neq 0)$$

are perpendicular. Find the coordinates of the point of intersection of the two lines.

2. Simplify

$$\frac{4x^{\frac{3}{2}} - x^{-\frac{1}{2}}}{2x^{\frac{1}{2}} - x^{-\frac{1}{2}}}, \quad x \neq 0, \ \frac{1}{2}.$$

Give your answer in the form $mx + n$ where m and $n \in \mathbf{N}$.

3. Given that

$$y = a\cos x - b\sin x$$

and that

$$\frac{dy}{dx} = 0 \quad \text{when} \quad x = \frac{\pi}{3},$$

find the value for a.
Give your answer in the form

$$\frac{mb}{\sqrt{n}}$$

where m and $n \in \mathbf{Z}$.

4. Let A and B be the matrices given by

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}.$$

Find a matrix M such that $M = BA^{-1}$.

5. Find the length of the tangent from the point $(-5, 8)$ to the circle

$$x^2 + y^2 - 4x - 6y + 3 = 0.$$

6. Given that $\sin\alpha = \frac{3}{5}$ and $\cos\beta = \frac{5}{13}$, $0 \leq \alpha \leq \frac{\pi}{2}$ and $0 \leq \beta \leq \frac{\pi}{2}$, find the value of

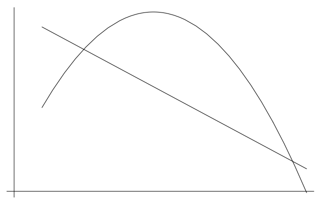
$$\tan(\beta - \alpha).$$

Give your answer in the form $\frac{m}{n}$ where $m,\, n \in \mathbf{Z}$.

7. The diagram shows the graphs of

$$y = -x^2 + 2x + 2 \quad \text{and} \quad y = -\frac{1}{2}\,x + 3.$$

Find the area enclosed between them.



8. If $2 + \sqrt{3}\,i$ is a root of a quadratic equation, with real coefficients, find the equation.
Give your answer in the form $az^2 + bz + c = 0$ where $a,\, b,\, c \in \mathbf{Z}$.

9. Given

$$y = \cos^4 x - \sin^4 x,$$

evaluate

$$\frac{dy}{dx}$$

when $x = \frac{7\pi}{12}$.

10. If α and β are the roots of the equation

$$3x^2 + 5x + 4 = 0,$$

find the value of

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha}.$$

Give your answer in the form $\frac{m}{n}$ where $m,\, n \in \mathbf{Z}$

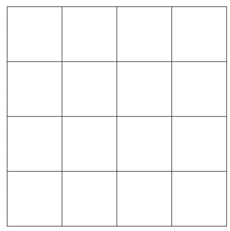
11. Find the term which is independent of x in the expansion of

$$\left(ax^2 - \frac{b}{x}\right)^9$$

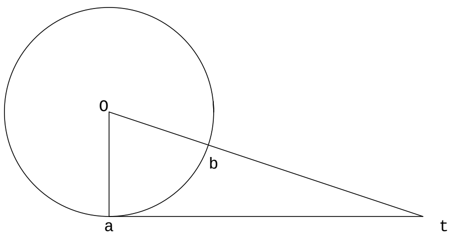
12. Solve
$$16^x - 5(2^{2x}) + 4 = 0.$$

13. Three indistinguishable coins are placed randomly, in different squares, on the 4×4 grid as shown.

    Calculate the probability that no two coins are in the same row or in the same column.



14. The diagram shows a circle with centre at o and radius r. The point a is on the circle and the line segment $[a\,t]$ is tangent to the circle at a. The line $[o\,t]$ intersects the circle at the point b and the angle $\theta = \widehat{aob} < \frac{\pi}{2}$. The area of the region bounded by the line segments $[a\,t]$, $[t\,b]$ and the arc ba of the circle is equal to half the area of the sector aob. Find an expression for $\tan\theta$ in terms of θ.



15. Four couples are to be seated at a round table. How many ways can this be done if the men and women are seated alternately.

16. Given that $\log_a x = b$, express $\log_a \frac{1}{x^2}$ in terms of b.

17. A bag contains 20 plastic strips whose lengths are each of the lengths 1 cm. to 20 cm. Three strips are chosen, without replacement, from the bag. The first strip is 8 cm. long and the second is 5 cm. long. What is the probability that the length of the third strip chosen is a number that could form a triangle with the lengths of the first two, given that the length of the third strip is a prime.

18. Find $\tan\theta$ where θ is the obtuse angle between the lines $3x - 2y + 1 = 0$ and $3x + 2y - 1 = 0$.
Give your answer in the form $\frac{m}{n}$ where $m, n \in \mathbf{Z}$.

19. If
$$f(x) = \frac{x}{x+1},$$
find
$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h}.$$

20. Find the equations of all circles which satisfy the following conditions.
The length of the radius is $\sqrt{20}$, the point $(-1, 3)$ lies on the circumference and the centre lies on the line $x + y = 0$.
Give your answer(s) in the form
$$(x-h)^2 + (y-k)^2 = (\text{radius})^2.$$