Team Quiz, Saturday, March 3rd, 2007

Be sure to answer exactly what is asked.

ROUND I

1. Find the range of values of $x \in \mathbb{R}$, $x \neq 3$, for which

$$\frac{x+2}{x-3} > 1.$$

Answer. If $x-3>0$ then $(x+2)/(x-3)>1$ if and only if $x+2>x-3$, which is always true.

If $x-3<0$ then $(x+2)/(x-3)>1$ if and only if $x+2<x-3$, which is never true.

Answer: $x > 3$

2. Given that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 23,$$

find x when $y = 2$.

Answer:

$$\begin{aligned} 3x^2 - 2xy + 2y^2 &= 23 \\ 3x^2 - 4x + 8 &= 23 \\ 3x^2 - 4x - 15 &= 0 \\ (3x+5)(x-3) &= 0. \end{aligned}$$

Answer: $x = 3, -5/3$

ROUND II

3. Find the value of the derivative of $\ln(\sec x)$ where $\sqrt{2}\cos(x) = 1$ and $0 < x < \pi/2$.

Answer.

$$\frac{d}{dx}(-\ln(\cos x)) = \frac{-1}{\cos x}(-\sin x) = \tan x.$$

Given $\cos(x) = 1/\sqrt{2}$, $\sin^2 x = 1/2$, and given the range, $\tan x = 1$.

Answer: 1

4. Evaluate

$$\int_0^1 \frac{x^3+1}{x+1}dx.$$

Answer.

$$\int_0^1 (x^2-x+1)dx = \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}.$$

Answer: 5/6

ROUND III

5. Find the positive root of

$$\left(1-\frac{1}{x}\right)^2 = 2,$$

and write it in the form $a+\sqrt{b}$ where $a, b \in \mathbb{Z}$.

Answer.

$$\begin{aligned} (x-1)^2 &= 2x^2 \\ 2x^2 - x^2 + 2x - 1 &= 0 \\ (x+1)^2 &= 2 \\ x = -1 \pm \sqrt{2}, \quad x &> 0: \end{aligned}$$

Answer: $-1+\sqrt{2}$

6. A person tosses a fair coin repeatedly until the sequence HH or TH is tossed. What is the probability that a person will stop tossing after tossing HH?

Answer. The only possible events are HH, HT...TH, or T...TH. Hence the answer is the probability that the first two tosses will be heads:

Answer: 1/4

ROUND IV

7. Solve for x, y, z (given that $x > 0, y > 0, z > 0$)

$$\begin{aligned} xy - 3x - 7y + 15 = 0 \quad &\therefore (x-7)(y-3) = 6 \\ xz - 2x - 7z + 8 = 0 \quad & \\ yz - 2y - 3z + 2 = 0 \quad & \end{aligned}$$

Answer.

$$\begin{aligned} (x-7)(y-3) &= 6 \\ (x-7)(z-2) &= 6 \\ (y-3)(z-2) &= 4 \\ y - 3 = z - 2 \qquad &\therefore y = z + 1 \\ (z+1-3)(z-2) = 4, \quad \text{so } z = 2 \pm 2, z > 0 \qquad &\therefore z = 4 \\ y = 5, (x-7)2 = 6, \quad &x = 10 \end{aligned}$$

Answer: $x = 10, y = 5, z = 4$

8. In a class test the average score for the girls is 91 and the average for the boys is 85. If the average score for the class is 89, what fraction of the class are boys?

Answer.

$$91g + 85b = 89(g+b) \qquad \therefore 2g = 4b.$$

The ratio of girls to boys is 2 : 1, and 1/3 of the class are boys.

Answer: 1/3

ROUND V

9. Evaluate

$$\int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{1 - \cos 2x} dx$$

Answer.

$$\int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{2\sin^2 x} dx = \frac{1}{2} \int_0^{\frac{\pi}{3}} \sin x dx = \frac{1}{2} [-\cos x]_0^{\frac{\pi}{3}} = \frac{1}{4}$$

Answer: 1/4

10. If

$$\frac{1+i}{1-i} = k\left(\frac{1-i}{1+i}\right), \quad i = \sqrt{-1},$$

find $k \in \mathbb{R}$.

Answer.

$$(1+i)^2 = k(1-i)^2, \quad 2i = k(-2i), \qquad \therefore k = -1.$$

Answer: -1

ROUND VI

11. Solve for y, where $y \in \mathbb{R}$.

$$2e^{2y} - 3e^y - 2 = 0$$

Answer. Let $x = e^y$.

$$2x^2 - 3x - 2 = 0 \qquad \therefore x = \frac{3 \pm \sqrt{9+16}}{4}.$$

Since y is real, only the positive root is possible, and $x = 2$.

Answer: $\ln 2$

12. A fair 12-sided die and a fair 8-sided die are rolled. What is the probability that the sum of the scores on the two dice is divisible by 5? Give the answer in the form

$$\frac{a}{b}, \quad a, b \in \mathbb{N}.$$

Answer. The total scores must be one of

$$5, 10, 15, 20.$$

These break down into

$$\begin{aligned} (1,4), (2,3), (3,2), (4,1), \\ \{(10-i, i) :\ 1 \le i \le 8\} \\ \{(15-i, i) :\ 3 \le i \le 8\} \\ (12, 8). \end{aligned}$$

Total: $4 + 8 + 6 + 1 = 19$, out of 96.

Answer: $19/96$

ROUND VII

13. Find the equation of the circle which touches the line $y = x$ at the point $(3, 3)$ and passes through the point $(5, 1)$.

Answer. Suppose the equation is $(x - a)^2 + (y - b)^2 = c^2$. Then

$$\begin{aligned} (5-a)^2 + (1-b)^2 &= c^2, \quad \text{so} \\ 25 - 10a + a^2 + 1 - 2b + b^2 &= c^2; \quad \text{again} \\ 9 - 6a + a^2 + 9 - 6b + b^2 &= c^2. \quad \text{Therefore} \\ 16 - 4a - 8 + 4b = 0 : &\quad a - b = 2. \end{aligned}$$

Also, the centre passes through the perpendicular to the tangent at $(3, 3)$, so

$$a + b = 6.$$

Therefore

$$a = 4, b = 2.$$

Then

$$c^2 = 1^2 + 1^2.$$

Answer: $(x - 4)^2 + (y - 2)^2 = 2.$

14. Find the value of $\tan(2\sin^{-1}(\frac{1}{3}))$. Give your answer in the form

$$\frac{a\sqrt{b}}{c}, \quad a, b, c \in \mathbb{N}.$$

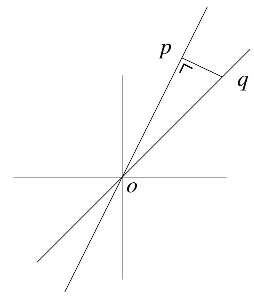
Answer. Write θ for $\sin^{-1}(1/3)$. $\tan(\theta) = (1/3)/(\sqrt{1 - 1/9}) = 1/(2\sqrt{2})$. We want

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{1/\sqrt{2}}{7/8} = \frac{4\sqrt{2}}{7}.$$

The answer is

Answer: $\frac{4\sqrt{2}}{7}$

15. The diagram shows two lines through the origin. Their slopes are 2 and 1. If $|op| = 1$, calculate $|pq|$.



Answer. Let α and β be the angles op and oq make with the positive x-axis, so $qop = \alpha - \beta$.

$$|pq| = |op| \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{2-1}{1+2} = \frac{1}{3}.$$

Answer: $|pq| = 1/3$.

16. Let $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $a \neq 0$, $-\pi < \theta < \pi$. Find $\frac{dy}{dx}$.

Answer.

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}.$$

The answer is

Answer: $\frac{\sin\theta}{1-\cos\theta}$

ROUND VIII

17. Evaluate

$$\int_3^4 \frac{2x-6}{x^2-6x+10}dx$$

Answer.

$$\int_3^4 \frac{(2x-6)dx}{x^2-6x+10} = \int_{x=3}^{x=4} \frac{d(x^2-6x+10)}{x^2-6x+10} = \int_1^2 \frac{du}{u} = \ln 2.$$

The answer is

Answer: $\ln 2$

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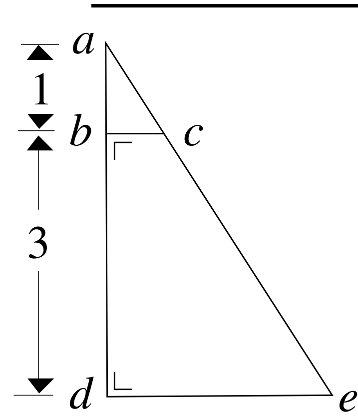
18. A right-angle triangle ade is divided as shown: $ab : bd = 1 : 3$. Find the probability that a point, chosen at random from the triangle ade, is in the region $bdec$.



Answer. By proportions, the large triangle has 16 times the area of the small one, so the probability is 15/16.

Answer: 15/16

19. Find the equations of the tangents to the circle

$$x^2 + y^2 - 4x + 6y - 3 = 0$$

from the point $(0, 1)$.

Answer.

$$x^2 + y^2 - 4x + 6y - 3 = 0$$
$$(x-2)^2 + (y+3)^2 = 16.$$

Length of the tangents is $(0-2)^2 + (1+3)^2 - 16 = 4$. Points of tangency are common to two circles.

$$\begin{aligned}(x-2)^2 + (y+3)^2 &= 16\\ x^2 + (y-1)^2 &= 4\\ -4x + 8y + 4 + 9 - 1 &= 16 - 4\\ x &= 2y\\ (2y)^2 + (y-1)^2 &= 4\\ (5y+3)(y-1) &= 0\\ (2,1), (-\frac{6}{5}, -\frac{3}{5})\end{aligned}$$

are the points of tangency. The equations are

$$y - 1 = \frac{1-1}{2-0}x, \quad y - 1 = \frac{1-(-3/5)}{0-(-6/5)}x$$

(The answers can be given in several forms.)

Answer: (a) $y = 1$; (b) $3y = 4x + 3$

20. The lengths of the sides of a triangle are 2cm, 3cm, and 4cm, respectively. Find the measure of the largest angle correct to the nearest degree.

Answer.

$$16 = 4 + 9 - 6\cos\theta, \qquad \cos\theta = -1/4, \qquad \theta = \cos^{-1}(-1/4) = 104.477512^\circ$$

Answer: $\theta \approx 104^\circ$