TEAM Maths 2005, Round I

(1) Evaluate $\int_0^3 x\sqrt{x+1}dx$. Express your answer in the form a/b, $a, b \in \mathbb{N}$.

Answer: Substitute $u = x + 1$:

$$\int_1^4 (u-1)\sqrt{u}du = \int_1^4 (u^{3/2} - u^{1/2})du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4 = \frac{64}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3} = \frac{116}{15}$$

(2) Given $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$, find an expression for c in terms of a and m.

Answer: Substitute for y in the circle equation to get points of intersection:

$$x^2 + (mx + c)^2 = a^2,$$

i.e.

$$x^2(1 + m^2) + 2mcx + c^2 - a^2 = 0.$$

So x is

$$\frac{-2mc \pm \sqrt{(2mc)^2 - 4(1+m^2)(c^2 - a^2)}}{2(1+m^2)}.$$

The line is a tangent iff this has repeated roots, i.e.,

$$4m^2c^2 - 4(1+m^2)(c^2 - a^2) = 0, \quad \text{i.e.} \quad -4c^2 + 4a^2(1+m^2) = 0,$$

so

$$c = \pm a\sqrt{1+m^2}.$$

TEAM Maths 2005, Round II

(1) Given $\log_a b + \log_b a^2 = 3, \quad a \neq b, \quad a, b > 0$, find b in terms of a.

Answer: Since $\log_a b = \log_e b / \log_e a$ etcetera, we get

$$\frac{\ln b}{\ln a} + 2\frac{\ln a}{\ln b} = 3.$$

Let $x = \ln a$ and $y = \ln b$, so

$$\frac{y}{x} + 2\frac{x}{y} = 3, \quad y^2 + 2x^2 = 3xy, \quad 2x^2 - 3xy + y^2 = 0:$$

$$y = \frac{3x \pm \sqrt{9x^2 - 8x^2}}{2} = \frac{3x \pm x}{2}, \quad \text{so} \quad y = x \text{ or } y = 2x.$$

$$\text{but } y \neq x, \quad \therefore y = 2x, \quad \therefore b = a^2.$$

(2) There are three bags marked A, B, and C. Each bag contains 4 discs numbered 1,2,3, and 4. A person chooses one disc at random from each bag. Let a, b, and c be the numbers on the discs chosen from A, B, and C, respectively. The person wins a prize when $a = b + c$. What is the probability the person wins a prize?

Answer: If $a = 1$, there are no suitable values of b, c. The combinations which result in a prize are (i) $a = 2, b = c = 1$, (ii) $a = 3, b = 1, c = 2$, (iii) $a = 3, b = 2, c = 1$, (iv) $a = 4, b = 1, c = 3$, (v) $a = 4, b = c = 2$, and (vi) $a = 4, b = 3, c = 1$.

The total probability is $3/32$.

TEAM Maths 2005, Round III

(1) Solve for x

$$4\left(16^{\sin^2 x}\right) = 2^{6\sin x}, \quad 0 \le x < 2\pi$$

Answer Divide by 4 and take the log of each side, to the base 2, getting

$$4\sin^2 x = 6\sin x - 2, \quad 4\sin^2 x - 6\sin x + 2 = 0.$$

$$\sin x = \frac{6 \pm \sqrt{36-32}}{8} = \frac{6 \pm 2}{8},$$

that is, $\sin x = 1/2$ or 1 and $x = \pi/6, \pi/2$, or $5\pi/6$.

(2) The colour of each face of a cube is chosen randomly and independently from the colours Red, Green and Blue. What is the probability that the cube has at least one pair of opposite faces which have the same colour?

Answer: First confine attention to one of the pairs of opposite faces. These get coloured in 9 possible ways of which 3 are monochromatic. Therefore the chance that the given pair is monochromatic is (1/3), and the chance that it is *not monochromatic* is $2/3$.

Considering all three pairs of opposite faces together, the chance that none is monochromatic is $(2/3)^3$, so the chance that at least one is monochromatic is $1 - (2/3)^3 = 19/27$.

TEAM Maths 2005, Round IV

(1) Given that

$$f(x) = 2^{kx} + 9$$

where $k \in \mathbb{R}$, if $f(3) : f(6) = 1 : 3$, find $f(9) - f(3)$.

Answer:

$$2^{6k} + 9 = 3(2^{3k} + 9).$$

Let $x = 2^{3k}$, so

$$x^2 - 3x - 18 = 0 : \quad (x-6)(x+3) = 0.$$

Since $k \in \mathbb{R}$, x is positive and $x = 6$. Then

$$f(9) - f(3) = 2^{9k} + 9 - 2^{3k} - 9 = x^3 - x = 210.$$

(2) Find the value of x^2 such that $\cos(\tan^{-1} x) = x$. Give your answer in the form

$$x^2 = \frac{a + \sqrt{b}}{c}, \quad a, b, c \in \mathbb{Z}.$$

Answer: Let $x = \tan\theta$, so $\cos(\tan\theta) = \tan\theta$. Square this and simplify, using $\cos^2\theta = 1/(1 + \tan^2\theta)$:

$$\frac{1}{1+x^2} = x^2, \quad x^2(1+x^2) = 1, \quad x^4 + x^2 - 1 = 0,$$

which becomes

$$x^2 = \frac{-1 \pm \sqrt{5}}{2},$$

which must be positive, so we choose the positive square root;

$$x^2 = \frac{\sqrt{5} - 1}{2}.$$

TEAM Maths 2005, Round V

(1) Suppose that

$$\frac{a}{b} = a^{3b} \quad \text{and} \quad ab = a^b,$$

where $a, b \in \mathbb{R}$, $a > 1$, $b > 0$, and $a > b$. Find a.

Answer: Multiply the equations together: $a^2 = a^{4b}$, so $4b = 2$ and $b = 1/2$. Substitute: $2a = a^{3/2}$, so $a^{1/2} = 2$ and $a = 4$.

(2) If $x + y = 4$ and $xy = -12$, what is the value of $x^2 + 5xy + y^2$?

Answer: $x^2 + 5xy + y^2 = (x + y)^2 + 3xy = -20$.

TEAM Maths 2005, Round VI

(1) There are 2 red, 2 black, 2 white, and a positive but unknown number of green socks in a drawer. If the probability of picking two socks of the same colour from the drawer, without replacement, is $1/5$, how many green socks are in the drawer?

Answer: Let n be the number of green socks. There are $\binom{n+6}{2}$ ways of choosing a pair, and of these choices there are $1+1+1+\binom{n}{2}$ pairs of the same colour. Therefore

$$\binom{n+6}{2} = 5\binom{n}{2} + 15.$$

$$(n+6)(n+5) = 5n(n-1)+30, \quad 5n^2-5n+30-n^2-11n-30=0, \quad \text{i.e. } 4n^2-16n=0.$$

Since n is positive, $n=4$.

(2) If x and y are real, solve for x and for y

$$x^2 - xy + 8 = 0; \quad x^2 - 8x + y = 0.$$

Answer: Subtract, so

$$8x - xy - y + 8 = 0; \qquad 8(x+1) = y(x+1),$$

so $x=-1$ or $y=8$. If $x=-1$ then $y=-9$, and if $y=8$ then $x^2-8x+8=0$, so $x=(8\pm\sqrt{32})/2$, or $x=4\pm2\sqrt{2}$. The solutions are

$$\begin{aligned} (x,y) &= \\ &(-1,-9), \\ &(4+2\sqrt{2},8), \\ \text{or} \quad &(4-2\sqrt{2},8). \end{aligned}$$

TEAM Maths 2005, Round VII

(1) Evaluate
$$\int_0^{3/\sqrt{2}} \frac{dx}{(9-x^2)^{3/2}}.$$

Answer: Substitute $x = 3\sin\theta$, getting
$$\int_0^{\pi/4} \frac{3\cos\theta d\theta}{\cos^3\theta} = \frac{1}{9}\int_0^{\pi/4} \sec^2\theta d\theta = \frac{1}{9}\left[\tan\theta\right]_0^{\pi/4} = \frac{1}{9}.$$

(2) A two-digit number is such that the square of its tens digit plus 10 times its units digit is equal to the square of its units digit plus 10 times its tens digit. The set of these numbers is written down. What is the probability that a number picked at random from this set is prime?

Answer: Writing t and u for its digits, $t^2 + 10u = u^2 + 10t$, so $t = u$ or $t + u = 10$. The possible numbers are $11, 22, 33, 44, 55, 66, 77, 88, 99$, and $19, 28, 37, 46, 55, 64, 73, 82, 91$. Of these 17 numbers, 4 are prime, so the answer is
$$\frac{4}{17}.$$

(3) The derivative of $f(x) - f(2x)$ at $x = 1$ is 5, and the derivative of $f(x) - f(2x)$ at $x = 2$ is 7.

Find the derivative of $f(x) - f(4x)$ at $x = 1$.

Answer:
$$f'(1) - 2f'(2) = 5; \quad f'(2) - 4f'(4) = 7$$
$$\therefore f'(1) - 4f'(4) = 5 + 2(7) = 19.$$

The answer is 19.

(4) When a certain quadratic expression is divided by $(x-1)$, $(x-2)$, and $(x-3)$, the remainders are $t, 2t$, and $4t$ respectively. Find, in terms of t, the remainder when the expression is divided by $(x-4)$.

Answer: by the Remainder Theorem, $f(1) = t$, $f(2) = 2t$, $f(3) = 4t$. Using Lagrange's interpolation method,
$$f(x) = t\frac{(x-2)(x-3)}{2} + 2t\frac{(x-1)(x-3)}{-1} + 4t\frac{(x-1)(x-2)}{2},$$
so
$$f(4) = t\frac{2}{2} + 2t\frac{3}{-1} + 4t\frac{6}{2} = 7t.$$

TEAM Maths 2005, Round VIII

(1) Given

$$\int_1^a \frac{x^4-1}{x^3}dx = \frac{9}{8},$$

where $a > 1$, find the value of a.

Answer:

$$\int_1^a \frac{x^4-1}{x^3}dx = \int_1^a \left(x - \frac{1}{x^3}\right)dx = \left[\frac{x^2}{2} + \frac{1}{2x^2}\right]_1^a; \quad \therefore \frac{a^2}{2} + \frac{1}{2a^2} - 1 = \frac{9}{8},$$

so

$$a^4 + 1 = \frac{17a^2}{4}; \quad a^2 = \frac{17/4 \pm 15/4}{2} = 4, 1/2.$$

Since $a > 1$, $a = 2$.

(2) Find the equations of the tangents from the point $(3, 2)$ to the circle $x^2 + y^2 + 4x + 6y + 8 = 0$.

Answer:

$$(x+2)^2 + (y+3)^2 = 5.$$

The centre is $(-2, -3)$.

First solution: we translate the centre of the circle, and the point $(3, 2)$, so the centre lies at $(0, 0)$. The translated point would be $(5, 5)$.

From Question 2, Round 1, if the centre had been $(0, 0)$ and $y = mx + c$ the equation of a tangent,

$$c = \pm\sqrt{5}\sqrt{1+m^2}.$$

Since $(5, 5)$ is on the two tangents, we substitute $y = mx + c$:

$$5 = 5m \pm \sqrt{5}\sqrt{1+m^2}$$

$$5(1-m) = \pm\sqrt{6}\sqrt{1+m^2}; \quad 25m^2 - 50m + 25 = 5 + 5m^2,$$

so $m^2 - \frac{5}{2}m + 1 = 0$. The solutions are $m = 2$ or $1/2$.

These slopes are unaffected by tranlating coordinates, so we can subsitute the original point $(3, 2)$ to get the tangents.

$$y - 2 = 2(x-3): \quad y - 2x + 4 = 0; \qquad \text{or} \quad y - 2 = \frac{x-3}{2}: \quad 2y - x - 1 = 0$$

are the equations of the tangents.

Alternative solution: Suppose (a, b) is a point of tangency from $(3, 2)$, so its displacement vector from $(3, 2)$ is orthogonal to its displacement vector from $(-2, -3)$. Therefore

$$(a+2)(a-3) + (b+3)(b-2) = 0; \quad a^2 - a + b^2 + b - 12 = 0. \qquad (*)$$

Substitute (a, b) into the equation of the circle, and subtract (*), getting

$$5a + 5b + 20 = 0; \quad b = -4 - a.$$

Substitute this into the equation of the circle, getting

$$(a+2)^2 + (-a-4+3)^2 = 5; \quad 2a^2 + 6a = 0.$$

Therefore $a = 0$ or $a = -3$. The contact points are therefore

$$(0, -4) \quad \text{and} \quad (-3, -1).$$

This gives two tangent lines:

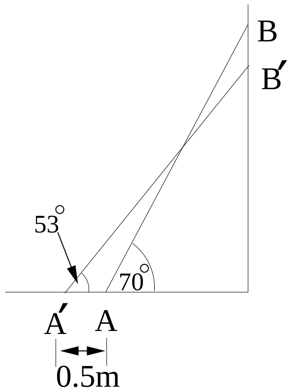
$$(y - (-4)) = \frac{2-(-4)}{3-0}(x - 0) : \quad 2x - y - 4 = 0,$$

and

$$(y - (-1)) = \frac{2-(-1)}{3-(-3)}(x - (-3)) : \quad x - 2y + 1 = 0.$$

(3)

A ladder AB is such that its foot is on horizontal ground and its top rests against a wall as shown. In this original position the ladder makes an angle of $70°$ as shown. The foot of the ladder is then moved 0.5m away from the wall, moving the ladder to the position A′B′. In this new position the ladder makes an angle of $53°$ with the horizontal. Calculate, to the nearest centimetre, how far the ladder slips down the wall, i.e. |BB′|.



Answer: If a is the distance of the foot of the ladder from the wall and b the distance of its top from the ground, c is its length, and y is the distance the top slips, then

$$a^2 \sec^2 70° = a^2 + b^2 = c^2,$$

and

$$(a + .5)^2 \sec^2 53° = (a + .5)^2 + (b - y)^2 = c^2.$$

Therefore

$$\frac{a + .5}{a} = \frac{\cos 53°}{\cos 70°} = 1.759527,$$

so $a = .658527$ metres, and $y = a \tan 70° - (a + .5) \tan 53° = .262806$ metres, or 26cm to the nearest centimetre.

(4) Each of the points $p(4,1)$, $q(7,-8)$, and $r(10,1)$ is a midpoint of a radius of the circle C. Calculate the radius of C.

Answer: Calculate the circumcentre s of the triangle pqr. By symmetry, its x-coordinate is 7. Also, it lies along the perpendicular bisector of p and q. This line passes through $(p+q)/2$ and has slope $1/3$, so it can be parametrised as $(5.5 + 3t, -3.5 + t)$. Setting $5.5 + 3t = 7$, $t = 1/2$, and $s = (7, -3)$; $q = (7, -8)$: so $|sq| = 5$. But $|sq|$ is half the radius. Therefore the radius is 10.